Functional Analysis

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1 Linear spaces and Linear operators

1.1 Linear spaces

Definition 1 (1.1) *Let X be a set and $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Assume that X is provided with two operations: addition and scalar multiplication, i.e., mappings from $X \times X$ to X and $\mathbb{K} \times X$ to X, denoted by*

$$(x, y) \mapsto x + y, \quad (\lambda, x) \mapsto \lambda x, \quad x, y \in X, \quad \lambda \in \mathbb{K}$$

respectively. Then X is said to be linear space over $\mathbb{K}$ if for all $x, y, z \in X$ and $\lambda, \mu \in \mathbb{K}$ the following axioms are satisfied:

(1) $x + y = y + x$;

(2) $(x + y) + z = x + (y + z)$;

(3) there exists an element $0 \in X$ *such that* $x + 0 = x$;

(4) There exists an element $-x \in X$ *such that* $x + (-x) = 0$;

(5) $\lambda(\mu x) = (\lambda\mu)x$;

(6) $1x = x$;

(7) $\lambda(x + y) = \lambda x + \lambda y$;

(8) $(\lambda + \mu)x + \lambda x + \mu x$

A subset V of a linear space X is called a linear subspace when it is a linear space itself with the given operations.

Remark: Note that in this course we often refer $\mathbb{K}$ to be either $\mathbb{R}$ or $\mathbb{C}$.

Examples:

$$\mathcal{F}(S, \mathbb{K}) = \{f : S \to \mathbb{K}\} \quad \mathcal{C}(S, \mathbb{K}) = \{f : [a, b] \to \mathbb{K} : f \text{ is continuous}\}$$

$$l^\infty = \{(x_1, x_2, ...) : x_i \in \mathbb{K}, \sup_{i \in \mathbb{N}} |x_i| < \infty\} \quad l^p = \{(x_1, x_2, ...) : x_i \in \mathbb{K}, \sum_{i=1}^{\infty} |x_i|^p < \infty\}$$

Definition 2 (1.5) *Let X be a linear space. The sum of two linear subspaces $V, W \subset X$ is defied as*

$$V + W = \{x + y : x \in V, y \in W\}$$

the sum is called direct if $V \cap W = \{0\}$

1.2 Linear operators

Definition 3 (Linear mapping) *Let X and Y be linear spaces over $\mathbb{K}$. A mapping $T : X \to Y$ is called linear if*

- $\operatorname{dom} T = X$
- $T(x+y) = Tx + Ty$
- $T(\lambda x) = \lambda(Tx)$

for all $x, y \in X$ and $\lambda \in \mathbb{K}$. The collection of all linear mappings from X to Y is denoted by $L(X, Y)$

Remark: Note that a linear map $T : X \to Y$ is injective iff $\ker T = \{0\}$ and is surjective iff $\operatorname{ran} T = Y$

Definition 4 (Projection) *Let X be a linear space and $P : X \to X$ be a linear mapping. Then P is called projection if $P^2 = P$.*

Lemma 5 (1.13) *A linear mapping $P : X \to X$ is a projection if and only if $I - P$ is a projection. In this case:*

$$\operatorname{ran} P = \ker(I - P), \quad \ker P = \operatorname{ran}(I - P)$$

Moreover, $X = \operatorname{ran} P + \ker P$ is a direct sum.

1.3 Quotient spaces of linear spaces

Definition 6 *A relation $\sim$ on a set X is called an equivalence relation if*

1. *reflexive: for each $x \in X$ one has $x \sim x$*
2. *symmetric: if $x \sim y$, then $y \sim x$*
3. *transitive: $(x \sim y \wedge y \sim z)$, then $x \sim z$*

Moreover, for $x \in X$ the equivalence class $[x]$ of x is defined as

$$[x] = \{y \in X : x \sim y\}$$

Definition 7 (Quotient set) *The set of all equivalence classes in X is denoted by $X/\sim$ (quotient set), and the mapping $\pi : X \to X/\sim$ given by $x \mapsto [x]$ is called the natural mapping*

Theorem 8 (1.18) *Let X be a set with an equivalence relation $\sim$. Let $x, y \in X$, then one has the following statements:*

1. $x \in [x]$
2. $[x] = [y] \Leftrightarrow x \sim y$
3. $[x] \cap [y] \neq \emptyset \Rightarrow [x] = [y]$
4. $X = \bigcup_{x \in X} [x]$, *the disjoint union of equivalence classes*

Definition 9 *Let X be a linear space and let $V \subset X$ be a linear subspace. Then V induces an equivalence relation on X by*

$$x \sim y \Leftrightarrow x - y \in V$$

The equivalence class to which $x \in X$ belongs is denoted by $x + V$

$$x + V = \{y \in X : x - y \in V\}$$

The set of equivalence classes is denoted by X/V

Definition 10 *Let X be a linear space and let $V \subset X$ be a linear subspace. The natural mapping $\pi : X \to X/V$ is defined by*

$$\pi(x) = x + V, \quad x \in X$$

moreover, the mapping π is linear, subjective and $\ker \pi = V$

1.4 Isomorphisms between linear spaces

Theorem 11 (Isomorphism Theorem) *let X, Y be linear spaces and let $T \in L(X; Y)$. Then, the map $\hat{T} : X/\ker T \to Y$, given by*

$$\hat{T}([x]) = T(x)$$

is well defined, linear and injective. As consequence, the spaces $X/\ker T$ and $\operatorname{ran} T$ are isomorphic. If in addition, T is surjective, $\hat{T} : X/\ker(T) \to Y$ is an isomorphism of linear spaces.

Theorem 12 *Let X be a linear space with $V \subset X$ a linear subspace. If $\dim X < \infty$, then* $\dim X/V < \infty$ *and*

$$\dim X/V = \dim X - \dim V$$

Corollary 13 *Let $T : X \to Y$ be a linear map with $\dim X < \infty$. Then*

$$\dim \ker T + \dim \operatorname{ran} T = \dim X$$

1.5 Dual spaces of linear spaces

Definition 14 *Let X be a linear space over $\mathbb{K}$. The dual space of X (algebraic dual) is defined as $X' = L(X, \mathbb{K})$. The elements of X' called functionals on X*

Lemma 15 *Let X be a finite-dimensional linear space. Then X' is a finite-dimensional linear space and* $\dim X' = \dim X$

Definition 16 (Bidual) *Let X be a linear space over $\mathbb{K}$. The second-dual space of X is defined as $X" = L(X', \mathbb{K})$. The natural mapping $J : X \to X"$ is given by*

$$J(x)(f) = f(x), \quad x \in X, \quad f \in X'$$

Lemma 17 *Let X be a finite-dimensional linear space. Then J is a bijection between X and $X"$*

2 Normed linear spaces and inner product spaces

2.1 Linear spaces with a norm

Definition 18 *Let X be a linear space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. The mapping $\|\cdot\| : X \to \mathbb{R}$ is called norm if for all $x, y \in X$ and $\lambda \in \mathbb{K}$ the following axioms are satisfied:*

1. $\|x\| \geq 0$
2. $\|x\| = 0 \Leftrightarrow x = 0$
3. $\|\lambda x\| = |\lambda| \|x\|$
4. $\|x + y\| \leq \|x\| + \|y\|$

The pair $(X, \|\cdot\|)$ is called a normed linear over $\mathbb{K}$. By abuse of language X itself will often be called a normed linear space. If in (2) only the implication ($\Leftarrow$) holds, then $\|\cdot$ is called a semi-norm and X is called a semi-normed linear space over $\mathbb{K}$.

Proposition 19 (Reverse Triangle Inequality) *For all $x, y \in X$, we have $|\|x\| - \|y\|| \leq \|x - y\|$*

Proposition 20 *The expression $d(x, y) = \|x - y\|$ defines a metric on X, and the function d is continuous on $X \times X$.*

Remark: This metric induce a natural topology, which we call the **strong topology** and is generated by the open balls

$$B(x_0, \epsilon) = \{y \in X : \|y - x_0\| < \epsilon\}, \quad \text{for } x_0 \in X \quad \text{and } \epsilon > 0$$

Proposition 21 (Young's inequality) *For $a, b \geq 0$, we have*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

Proposition 22 (Holder's inequality) *For $x \in \mathbb{K}^n$ and let $1/p + 1/q = 1$, where $1 < p < \infty$, we have*

$$\sum_{i=1}^n |x_i y_i| \leq \left(\sum_{i=1}^n |x_i|^p\right)^{1/p} \left(\sum_{i=1}^n |y_i|^q\right)^{1/q}$$

Proposition 23 (Minkowski's Inequality) *For $x \in \mathbb{K}^n$ and let $1 \leq p < \infty$, we have*

$$\left(\sum_{i=1}^n |x_i + y_i|^p\right)^{1/p} \leq \left(\sum_{i=1}^n |x_i|^p\right)^{1/p} + \left(\sum_{i=1}^n |y_i|^p\right)^{1/p}$$

Definition 24 *A sequence (x_n) in a normed space X converges to $x \in X$ if*

$$\|x_n - x\| \to 0 \quad \text{as } n \to \infty$$

In other words, for every $\epsilon > 0$, there is $N \in \mathbb{N}$, such that

$$\|x_n - x\| \leq \epsilon$$

for all $n \geq N$

Proposition 25 *If $x_n \to x$ in X, then $\|x_n\| \to \|x\|$ in $\mathbb{R}$. As a consequence, convergent sequences are bounded.*

Proposition 26 (Topological vector spaces) *The sum and the multiplication by a scalar are continuous functions. More precisely, if $x_n \to x$ in X, $y_n \to y$ in X and $\lambda_n \to \lambda$ in $\mathbb{K}$, then*

$$x_n + y_n \to x + y \quad \text{and} \quad \lambda_n x_n \to \lambda x$$

in X. Normed spaces are topological vector spaces

Definition 27 (distance) *Let X be a normed space. The distance between a point $x \in X$ and a set $S \subset X$ is*

$$d(x, S) = \inf\{\|x - y\| : y \in S\}$$

It is a continuous function on X

Definition 28 *The closure of $S \subset X$, where X is a normed space is*

$$\overline{S} := \{x \in X : d(x, S) = 0\}$$

Proposition 29

- *A point $x \in X$ is in $\overline{S}$ if, and only if, a sequence in S converges to x*
- *A set $S \subset X$ is closed if, and only if, $\overline{S} = S$ (or $\overline{S} \subset S$)*
- *If V is a closed subspace, then $\|[x]\| := d(x, V)$ is a norm in X/V*
- *Every subspace of finite dimension is closed*

Definition 30 *Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on X are equivalent if there exist $m, M > 0$ such that*

$$m\|x\|_1 \leq \|x\|_2 \leq M\|x\|_1$$

for all $x \in X$

Proposition 31 *Equivalent norms induce the same topology: they have the same open sets and the same convergent sequences.*

Theorem 32 *If $\dim X < \infty$, all norms on X are equivalent*

Remark: This is not true in infinite-dimensional spaces!

Theorem 33 *The closed unit ball in a normed space X is compact if and only if $\dim X < \infty$*

Lemma 34 (Riesz's Lemma) *Let V be a closed linear subspace of a normed space X with $V \neq X$ and let $0 < \lambda < 1$. Then, there is $x_\lambda \in X$ such that $\|x_\lambda\| = 1$ and $\|x_\lambda - v\| > \lambda$ for all $v \in V$.*

3 Banach spaces

3.1 Banach spaces

Definition 35 *A sequence (x_n), in a normed space X, has the Cauchy property (or is a Cauchy sequence) if*

$$\|x_n - x_m\| \to 0 \quad as \quad n, m \to \infty$$

More precisely, for every $\epsilon > 0$, there is $N \in \mathbb{N}$ such that $\|x_n - x_m\| \leq \epsilon$ for all $n, m \geq N$

Proposition 36 *Every convergent sequence has the Cauchy property*

Proposition 37 *Every Cauchy sequence is bounded and has, at most, one limit point*

Proposition 38 *A Cauchy sequence with a convergent subsequence must be convergent.*

Definition 39 *A normed linear space $(X, \|\cdot\|)$ is called complete if every Cauchy sequence in X converges in X*

Definition 40 (Banach Space) *A Banach space is a normed space in which every Cauchy sequence is convergent*

Proposition 41 *Let X be a finite-dimensional normed linear space. Then X is a Banach space*

Proposition 42 *Let V be a subspace of a normed space X. We have the following:*

1. *If X is a Banach space and V is closed, then V is a Banach space*
2. *If V is a Banach space, then V is closed in X*

Theorem 43 *Let X be a normed space. The following are equivalent:*

- *X is a Banach space*
- *Every absolutely convergent series is convergent*

Theorem 44 *If V is a closed subspace of a Banach space X, then the quotient space X/V is a Banach space*

Theorem 45 *For each normed space X there exist a Banach space $\mathbb{X}$ and a linear isometry $\iota : X \to \mathbb{X}$ such that $\overline{\iota(X)} = \mathbb{X}$*

4 Baire's Theorem, Bounded Linear Operators and Uniform Boundedness Principle, Open Mapping Theorem

Theorem 46 (Open Mapping Theorem) *Let X and Y be Banach spaces. If $T \in L(X, Y)$ is bounded and surjective, then T is open: it maps open subsets of X to open subsets of Y*

Theorem 47 (Bounded Inverse Theorem) *Let X and Y be Banach spaces, let $T \in L(X, Y)$ be bounded and bijective. Then T^{-1} is bounded*

Theorem 48 (Closed Range Theorem) *Let X and Y be Banach spaces, and let $T \in L(X,Y)$ be bounded. The following statements are equivalent:*

1. *There is $c > 0$ such that $\|Tx\| \geq c\|x\|$ for all $x \in X$;*
2. *T is injective and $\operatorname{ran}(T)$ is closed in Y*

Theorem 49 (Equivalence of Banach Norms) *Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on a linear space X, both of which make X a Banach space. Assume there is a constant $C > 0$ such that*

$$\|x\|_2 \leq C\|x\|_1$$

for all $x \in X$. Then, there is a constant $C' > 0$ such that

$$\|x_1\| \leq C'\|x\|_2$$

for all $x \in X$. As a consequence, the two norms are equivalent

Definition 50 *Let X and Y be normed spaces, and let V be a closed subspace of X. An operator $T \in L(V,Y)$ is closed if its graph*

$$G(T) = \{(x, Tx) : x \in V\}$$

is closed subset of $X \times Y$

Theorem 51 (Closed Graph Theorem) *Let X and Y be normed spaces, let V be a closed subspace of X*

1. *If $T \in L(V,Y)$ is bounded, it is closed*
2. *If X and Y are Banach spaces and $T \in L(V,Y)$ is closed, then T is bounded*

Definition 52 (Bounded projection) *Let V and W be closed linear subspaces of a Banach space X. Assume*

$$X = V + W \quad \text{with} \quad V \cap W = \{0\}$$

so that each $x \in X$ is uniquely written as $x = v + w$, with $v \in V$ and $w \in W$. Define the projection operator $P : X \to X$ by $Px = v$, for each $x \in V$. Then P is bounded.

Theorem 53 *Let T be a bounded linear operator on a Banach space X. If $\|T\| < 1$, then $I - T$ is invertible: there is a bounded linear operator S on X such that $S(I-T) = (I-T)S = I$*

5 Hahn-Banach Theorem(s)

Definition 54 *Given a linear space X, its algebraic dual is the space $X' = L(X, \mathbb{K})$.*

Definition 55 *Given a normed space X, its topological dual is the space $X^* = \mathcal{L}(X, \mathbb{K})$*

Remark: Its elements are bounded linear functionals on X.

Remark: Since $\mathbb{K} \in \{\mathbb{C}, \mathbb{R}\}$, X^* is always a Banach space, even if X is not.

Remark: Let $\|\cdot\|_*$ be the norm on X^*

Definition 56 *The bilinear function* $\langle \cdot, \cdot \rangle : X^* \times X \to \mathbb{K}$*, defined by* $\langle L, x \rangle = L(x)$ *is the duality product between* X *and* X^**. If the spaces are not clear from the context, we write* $\langle L, x \rangle_{X^*,X}$

Theorem 57 (Hahn-Banach Separation Theorem) *Let* A *and* B *be nonemprty, disjoint convex subsets of a normed space* X*.*

- *If* A *is open, there exists* $L \in X^* \backslash \{0\}$ *such that* $\langle L, x \rangle < \langle L, y \rangle$ *for each* $x \in A$ *and* $y \in B$
- *If* A *is compact and* B *is closed, there exist* $L \in X^* \backslash \{0\}$*, and* $\epsilon > 0$ *such that* $\langle L, x \rangle + \epsilon \leq \langle L, y \rangle$ *for each* $x \in A$ *and* $y \in B$

Proposition 58 *Given* $N \geq 1$*, let* C *be a nonempty and convex subset of* $\mathbb{R}^N$ *not containing the origin. Then, there exists* $v \in \mathbb{R}^N \backslash \{0\}$ *such that* $v \cdot x \leq 0$ *for each* $x \in C$*. In particular, if* $N \geq 2$ *and* C *is open, then*

$$V = \{x \in \mathbb{R}^N : v \cdot x = 0\}$$

is a nontrivial subspace of $\mathbb{R}^N$ *that does not intersect* C

Corollary 59 (of HBST) *For each* $x \in X$*, there is* $l_x \in X^*$ *such that* $\|l_x\|_* = 1$ *and* $\langle l, x \rangle = \|x\|$*. We say* l_x *is a support functional at* x

Corollary 60 *For every* $x \in X$*,* $\|x\| = \max_{\|L\|_*=1} \langle L, x \rangle$

Theorem 61 (Hahn-Banach Extension Theorem) *Let* V *be a subspace of* X *and let* $l \in V^*$ *with* $\|l\|_{V^*} \leq \alpha$*, with* $\alpha > 0$*. Then, there exists* $L \in X^*$ *such that*

- L *coincides with* l *on* V
- $\|L\|_{X^*} \leq \alpha$

Theorem 62 *Let* X *be a normed linear space, let* $V \subset X$ *be a linear subspace, and let* $x_0 \in X$*. Assume that*

$$\delta = d(x_0, V) = \inf\{\|x_0 - v\| : v \in V\} > 0$$

Then there exists $F \in X^*$ *such that*

$$F(x_0) = \delta, \quad F \upharpoonright V = 0, \quad \text{and } \|F\| = 1$$

Corollary 63 *every* $l \in V^*$ *is the restriction to* V *of some* $L \in X^*$

Corollary 64 *if* $x_0 \notin V$*, there exists* $L \in X^*$ *such that* $L = 0$ *on* V *and* $\langle L, x_0 \rangle = 1$

Definition 65 (Duality mapping) *The (normalized) duality mapping is the set-valued function* $\mathcal{F} : X \to \mathcal{P}(X^*)$ *given by*

$$\mathcal{F}(x) = \{x^* \in X^* : \|x^*\|_* = 1 \text{ and } \langle x^*, x \rangle = \|x\|\}$$

The set $\mathcal{F}(x)$ *is always convex, but it need not be a singleton*

Definition 66 (Bidual) *The bidual of X is the dual of $X^* : X^{**} = \mathcal{L}(X^*, \mathbb{K})$*

Definition 67 (Evaluation functional) *For each $x \in X$, we define the evaluation functional $\mu_x : X \to \mathbb{R}$ by*

$$\mu_x(L) = \langle L, x \rangle$$

for each $L \in X^*$

Proposition 68 *For each $x \in X$ and $L \in X^*$, we have $\mu_x \in X^{**}$ and $\|\mu_x\|_{**} \leq \|x\|$*

Definition 69 *The linear function $\mathcal{J} : X \to X^{**}$, defined by $\mathcal{J}(x) = \mu_x$, is the canonical embedding of X into X^{**}*

Proposition 70 *The canonical embedding $\mathcal{J} : X \to X^{**}$ is an isometry.*

Definition 71 *The space X is reflexive if $\mathcal{J}$ is surjective: if every element μ of X^{**} is of the form $\mu = \mu_x$ for some $x \in X$*

Remark: Every reflexive space is a Banach space

Proposition 72 *A Banach space X is reflexive if, and only if, X^* is reflexive*

Proposition 73 *If V is a closed subspace of a reflexive space X, then V is reflexive*

Definition 74 *A subspace E of a normed space X is dense if $\overline{E} = X$*

Definition 75 (Separable) *A normed space X is separable if it contains a countable subset which is dense in it*

Remark: Every finite dimensional normed space is separable.

Remark: The space $\updownarrow^{\infty}$ is separable whenever $1 \leq p < \infty$

Theorem 76 *If X' is separable, so is X*

Corollary 77 *X is separable and reflexive if, and only if, X' is separable and reflexive. Finite dimensional spaces are separable and reflexive*

6 Weak Topology

Definition 78 (Neighborhood) *A neighborhood of $x \in X$ is a set $N \subset X$ such that there is $\mathcal{O} \in \mathcal{T}$ with*

$$x \in \mathcal{O} \subset N$$

The collection of all neighborhoods of $x \in X$ is denoted by $\mathcal{N}_{\mathcal{T}}(x)$, where we omit the $\mathcal{T}$ if it is either clear from the context or not relevant.

Theorem 79 *A set S is open if, and only if, $S \in \mathcal{N}(x)$ for all $x \in S$ (it is a neighborhood of each of its points)*

Definition 80 *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on a set X. If $\mathcal{T}_1 \subset \mathcal{T}_2$ we say that $\mathcal{T}_1$ is coarser than (or equal to) $\mathcal{T}_2$ and $\mathcal{T}_2$ is finer than $\mathcal{T}_1$*

Remark: The intersection of all topologies containing a family $\mathcal{S}$ of subsets of X is a topology. It is the coarsest topology containing $\mathcal{S}$. This topology must also contain the finite intersections of members of $\mathcal{S}$

Definition 81 (Basis for a topology) *A basis for the topology $\mathcal{T}$ is a subset $\mathcal{B}$ of $\mathcal{T}$ such that for every $x \in X$ and every $N \in \mathcal{N}(x)$, there is $B \in \mathcal{B}$ satisfying*

$$x \in B \subset N$$

Proposition 82 *If $\mathcal{B}$ is a basis for $\mathcal{T}$, then $\mathcal{T}$ is the coarsest topology containing $\mathcal{B}$. Therefore, bases uniquely determine the topology. The finite intersections of members of $\mathcal{S}$ are a basis for the coarsest topology containing $\mathcal{S}$*

Definition 83 (Continuity) *Let $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ be topological spaces. A function $f : X \to Y$ is continuous at a point $x \in X$ if $f^{-1}(N) \in \mathcal{N}_{\mathcal{T}}(x)$ for every $N \in \mathcal{N}_{\mathcal{S}}(f(x))$. It is continuous in X if it is so at every $x \in X$.*

Definition 84 *The weak topology on X is the weakest/coarsest/ fewest open sets topology on X that makes $f \in X'$ continuous.*

Remark: The weak topology is the coarsest topology containing the half-spaces.

Definition 85 *The finite intersections of half-spaces are a basis for the weak topology.*

Definition 86 (Neighborhood basis) *For a $\epsilon > 0$, $x_0 \in X$ and $f_1, ..., f_n \in X'$*

$$\bigcap_{k=1}^{n} \{x \in X \, | f_k(x - x_0)| < \epsilon\}$$

Remark: the weak topology is not metrializeble.

Definition 87 *A topological space has the Hausdorff property if distinct points admit disjoint neighborhoods: if $x \neq y$, there exists $N_x \in \mathcal{N}(x)$ and $N_y \in \mathcal{N}(y)$ such that $N_x \cap N_y = \emptyset$*

Proposition 88 *Every normed space with the weak topology has the Hausdorff property.*

Definition 89 *The weak topology in X' is then the coarsest topology fro which all the elements of X" are continuous*

Definition 90 *The weak* topology in X' is the coarsest topology that makes all the elements of $\mathcal{J}(X) \subset X''$ continuous. Recall that all the elements of $\mathcal{J}(X)$ are the evaluation functionals.*

Theorem 91 (Weierstrass) *Let C be compact, and let $f : C \to \mathbb{R}$ be continuous. Then, f attains its minimum and its maximus on C*

Theorem 92 *Let X be a Banach space.*

- *The closed balls in X' are weak*ly compact (Banach-Alaoglu)*
- *If X is separable, then the closed balls in X' are also weak*ly sequentially compact, which means that every bounded sequence has a weak*ly convergent subsequence.*

Theorem 93 *Let X be a Banach space. The following statements are equivalent:*

1. *X is reflexive*
2. *The closed balls in X are weakly compact*
3. *The closed balls in X are weakly sequentially compact, which means that every bounded sequence has a weakly convergent subsequence*

6.1 some application

Theorem 94 *Let X be reflexive, and let $f : X \to \mathbb{R}$ be a continuous convex function such that $\lim_{\|x\|\to\infty} f(x) = \infty$. Then, f attains its minimum.*

7 Hilbert spaces

Definition 95 (Inner product) *An inner product in a vector space H is a function $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{K}$ such that*

1. *$\langle x, x \rangle > 0$ for every $x \neq 0$*
2. *$\langle x, y \rangle = \overline{\langle y, x \rangle}$ for each $x, y \in H$*
3. *$\langle \alpha x + y, z \rangle = \alpha \langle x, z \rangle + \langle y, z \rangle$ for each $\alpha \in \mathbb{K}$ and $x, y, z \in H$*

Remark: $\langle x, \alpha y + z \rangle = \overline{\alpha} \langle x, y \rangle + \langle x, z \rangle$ for each $\alpha \in \mathbb{K}$ and $x, y, z \in H$

Definition 96 *The function $\| \cdot \| : H \to \mathbb{R}$, defined by $\|x\| = \sqrt{\langle x, x \rangle}$, is a norm on H.*

Proposition 97 *For each $x, y \in H$ we have*

- ***The Cauchy-Schwarz inequality:*** $| \langle x, y \rangle | \leq \|x\| \|y\|$
- ***Triangle inequality:*** $\|x + y\| \leq \|x\| + \|y\|$

Definition 98 *Given $y \in H$, define a function $L_y : H \to \mathbb{K}$ by $L_y(h) = \langle h, y \rangle$*

Proposition 99 *The function $\mathcal{L} : H \to H^*$, defined by $\mathcal{L}(y) = L_y$, is an isometry*

Proposition 100 *Let (x_n) and (y_n) be sequences in H. If $x_n \to x$ and $y_n \to y$, then*

$$\lim_{n\to\infty} \langle x_n, y_n \rangle = \langle x, y \rangle$$

Definition 101 *We say x, y are orthogonal, and write $x \perp y$, if $\langle x, y \rangle = 0$*

Theorem 102 (Pythagoras) *If $x \perp y$, then $\|x + y\|^2 = \|x\|^2 + \|y\|^2$*

Proposition 103 (Parallelogram identity) *For each $x, y \in H$, we have $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$*

Definition 104 *If $\|x\| = \sqrt{\langle x, x\rangle}$ for all $x \in X$, we say that the norm $\|\cdot\|$ is associated to the inner product $\langle \cdot, \cdot \rangle$*

Definition 105 (Hilbert space) *A Hilbert space is a Banach space, whose norm is associated to an inner product*

7.1 Orthogonal projection

Proposition 106 *Let K be a nonempty, closed and convex subset of H and let $x \in H$. Then, there exists a unique $y^* \in K$ such that*

$$\|x - y^*\| = \min_{y \in K} \|x - y\|$$

Moreover, it is the only element of K such that

$$\langle x - y^*, y - y^* \rangle \leq 0 \quad \forall y \in K$$

Remark: The point y^* is the projection of x onto K and will be denoted by $P_K(x)$

Proposition 107 *Let K be a nonempty, closed and convex subset of H. The function $x \mapsto P_K(x)$ is non-expansive*

Proposition 108 *If M is closed subspace of H, then $x - P_M(x) \perp M$ for each $x \in H$*

7.2 Representation Theorem

Recall that each $y \in H$ defines $L_y \in H^*$ by $L_y(h) = \langle h, y\rangle$, moreover $\|L_y\|_* = \|y\|$

Theorem 109 (Riesz-Frechet) *For each $L \in H^*$, there is a unique $y_L \in H$ such that*

$$L(h) = \langle y_L, h\rangle$$

for each $h \in H$. Therefore, the function $L \mapsto y_L$ is an isometric isomorphism.

Corollary 110 *The inner product $\langle \cdot, \cdot \rangle_* : H^* \times H^* \to \mathbb{K}$, defined by*

$$\langle L_1, L_2 \rangle_* = L_1(y_{L_2}) = \langle y_{L_1}, y_{L_2}\rangle$$

turns H^* into a Hilbert space, which is isometrically isomorphic to H. The norm associated with $\langle \cdot, \cdot \rangle_*$ is precisely $\|\cdot\|_*$

Corollary 111 *Hilbert spaces are reflexive*

Remark: A sequence (x_n) on a Hilbert space H converges weakly to $x \in H$ if, and only if, $\lim_{n\to\infty} \langle x_n - x, y\rangle = 0$ for all $y \in H$

Proposition 112 *A sequence in (x_n) converges strongly to x if, and only if, it converges weakly to x and $\lim_{n\to\infty} \sup \|x_n\| \leq \|x\|$*

7.3 Orthonormalization

Definition 113 (Orthonormal sets) *A set $\{e_i\}_{i \in I}$ in a linear space H with an inner product is orthonormal if*

- $\|e_i\| = 1$ *for all* $i \in I$*; and*
- $\langle e_i, e_j \rangle = 0$*, whenever* $i \neq j$

Remark: Every finite subset of an orthonormal set is linearly independent

Proposition 114 *Let $\{e_1, ..., e_n\}$ be an orthonormal set in a linear space H with an inner product, and let $V = \text{span}\{e_1, ..., e_n\}$. Then, for every $x \in H$*

$$P_V(x) = \sum_{i=1}^{n} \langle x, e_i \rangle \, e_i$$

Proposition 115 (Gram-Schmidt) *Given a linearly independent set $\{a_1, ..., a_n\}$ in a linear space H with an inner product, there is an orthonormal set $\{e_1, ..., e_n\}$ such that*

$$\text{span}\{e_1, ..., e_n\} = \text{span}\{a_1, ..., a_k\}$$

for all $k \in \{1, ..., n\}$

Procedure: (also for the proof)

1. Define
$$e_1 = \frac{a_1}{\|a_1\|} \Rightarrow \|e_1\| = 1$$
note that if $n = 1$, then the Gram-Schmidt is indeed true, $\text{span}\{e_1\} = \text{span}\{a_1\}$.

2. if $n > 1$, then define recursively
$$e_{n+1} = \frac{b_{n+1}}{\|b_{n+1}\|}, \quad b_{n+1} = a_{n+1} - P_{V_n}(a_{n+1})$$
where $V_n = \text{span}\{e_1, ..., e_n\} = \text{span}\{a_1, ..., a_n\}$

3. The vectors $\{e_1, ..., e_{n+1}\}$ are orthonormal and $\text{span}\{e_1, ..., e_{n+1}\} = \text{span}\{a_1, ..., a_{n+1}\}$

Proposition 116 *Let $\{e_i\}_{i \in \mathbb{N}}$ be an orthonormal set in a linear space H with an inner product. Then, for every $x \in H$,*

$$\sum_{i=1}^{\infty} | \langle x, e_i \rangle |^2 \leq \|x\|^2$$

Proposition 117 *Let $\{e_i\}_{i \in \mathbb{N}}$ be an orthonormal set in Hilbert space H. The series $\sum_{i=1}^{\infty} \lambda_i e_i$ is convergent if, and only if, $\sum_{i=1}^{\infty} |\lambda_i|^2 < \infty$. In such case,*

$$\left\| \sum_{i=1}^{\infty} \lambda_i e_i \right\|^2 = \sum_{i=1}^{\infty} |\lambda_i|^2$$

Definition 118 *An orthonormal set $\{e_i\}_{i\in\mathbb{N}}$ is an orthonormal basis for a Hilbert space H if*

$$\overline{\text{span}}\{e_i\}_{i\in\mathbb{N}} = H$$

Theorem 119 *A Hilbert space is separable if, and only if, it has an orthonormal basis*

Theorem 120 *The following statements about orthonormal set $\{e_i\}_{i\in\mathbb{N}}$ in a Hilbert space H are equivalent:*

1. $\{e_i\}_{i\in\mathbb{N}}$ *is an orthonormal basis for H*
2. $\{e_i\}_{i\in\mathbb{N}}^{\perp} = \{0\}$
3. *For each $x \in H$, $\sum_{i=1}^{\infty} \langle x, e_i\rangle\, e_i = x$*
4. *For each $x \in H$, $\sum_{i=1}^{\infty} |\langle x, e_i\rangle|^2 = \|x\|^2$*

8 Adjoint operators

Definition 121 *Let X and Y be Hilbert spaces, and let $T \in B(X,Y)$. The adjoint of T is the bounded linear operator $T^* \in B(Y,X)$ satisfying*

$$\langle Tx, y\rangle = \langle x, T^*y\rangle$$

for all $x \in X$ and $y \in Y$

Theorem 122 *The adjoint of L is well defined, unique, and has the following properties:*

1. $(T^*)^* = T$
2. $\|T^*\| = \|T\|$
3. $\|T^*T\| = \|T\|^2$

Remark: If T is a matrix, then $T^* = \overline{T}^t$

Proposition 123 *Let X,Y and Z be Hilbert spaces. We have the following*

- *If $T, S \in B(X,Y)$ and $\lambda, \mu \in \mathbb{K}$, then $(\lambda T + \mu S)^* = \overline{\lambda}T^* + \overline{\mu}S^*$*
- *If $T \in B(X,Y)$ and $S \in B(Y,Z)$, then $(ST)^* = T^*S^*$*

Proposition 124 *Let T be invertible. Then, T^* is invertible and $(T^*)^{-1} = (T^{-1})^*$*

Lemma 125 *For $T \in B(X,Y)$ and $\lambda \in \mathbb{K}$ we have*

$$(\text{ran}(T-\lambda))^{\perp} = \ker(T^* - \overline{\lambda}) \subset Y$$
$$(\text{ran}(T^*-\overline{\lambda}))^{\perp} = \ker(T - \lambda) \subset X$$

Corollary 126 *Given $T \in B(X)$ and $\lambda \in \mathbb{K}$, we have the following decompositions:*

$$X = \ker(T^* - \overline{\lambda}) \oplus \overline{ran}(T - \lambda)$$
$$X = \ker(T - \lambda) \oplus \overline{ran}(T^* - \overline{\lambda})$$

where $\oplus$ denotes a direct sum of orthogonal closed subspaces.

Definition 127 *Let X be a Hilbert space. An operator $T \in B(X)$ is selfadjoint if $T^* = T$.*

Definition 128 *Let X be a Hilbert space. An operator $T \in B(X)$ is normal if $TT^* = T^*T$*

Proposition 129 *Let X be a Hilbert space.*

- *If T is selfadjoint, it is normal*
- *If $T \in B(X)$ is normal, then*
  - $\|T^*x\| = \|Tx\|$ *for all $x \in X$*
  - $\ker(T - \lambda) = \ker(T^* - \overline{\lambda})$ *for all $\lambda \in \mathbb{K}$*

Definition 130 *Let X be a Hilbert space. An operator $P \in B(X)$ is an orthogonal projection if:*

- $P^2 = P$
- $\ker P \perp \operatorname{ran} P$

Proposition 131 *A projection P in a Hilbert space H is orthogonal if, and only if, it is selfadjoint*

9 Eigenvalues, Eigenvectors of linear operators

Definition 132 *Let X be a Banach space, and let $T \in B(X)$. A scalar $\lambda \in \mathbb{K}$ is an eigenvalue of T if there is $x \neq 0$ such that*

$$Tx = \lambda x$$

The space $\ker(T-\lambda)$ is the associated eigenspace, and its nonzero elements are the eigenvectors of T

Proposition 133 *λ is an eigenvalue of T if, and only if, $\overline{\lambda}$ is an eigenvalue of T^*.*

Proposition 134 *If H is a Hilbert space and $T \in B(H)$ is normal, then the eigenspaces corresponding to distinct eigenvalues are orthogonal to each other.*

Definition 135 *Let X be a Banach space and let $T \in B(X)$. The resolvent set of T is*

$$\rho(T) = \{\lambda \in \mathbb{K} : (T - \lambda)^{-1} \in B(X)\}$$

The resolvent operator of T with index $\lambda \in \rho(T)$ is $R(\lambda) = (T - \lambda)^{-1}$

Definition 136 *Let X be a Banach space and let $T \in B(X)$. The spectrum of T is $\sigma(T) = \mathbb{K} \setminus \rho(T)$*

Remark: $\rho(T^*) = \overline{\rho(T)}$ and $\sigma(T^*) = \overline{\sigma(T)}$

Proposition 137 *Let X be a Banach space, and let $T \in B(X)$. If $\lambda \in \sigma(T)$, then $|\lambda| \leq \|T\|$. In turn, if $|\lambda| > \|T\|$, then $\lambda \in \rho(T)$ and*

$$R(\lambda) = -\sum_{n=0}^{\infty} \frac{T^n}{\lambda^{n+1}}$$

Proposition 138 *Let H be a Hilbert space, and let $T \in B(H)$ be normal. Then, $\lambda \in \rho(T)$ if, and only if, there is $c > 0$ such that*

$$\|(T - \lambda)x\| \geq c\|x\|$$

for all $x \in H$
As a consequence, $\lambda \in \sigma(T)$ if, and only if, there is a sequence (x_n) such that $\|x_n\| = 1$ for all n, and $(T - \lambda)x_n \to 0$

10 Compact operators

Corollary 139 *Let X be a Banach space, and let $T \in B(X)$. Then $\rho(T)$ is open and $\sigma(T)$ is closed.*

Definition 140 *Let X and Y be Banach spaces. A linear operator $T : X \to Y$ is compact if $\overline{T(B)}$ is compact whenever $B \subset X$ is bounded*

Proposition 141 *$T : X \to Y$ is compact if, and only if, for every bounded sequence (x_n), the sequence (Tx_n) has a convergent subsequence*

Proposition 142 *Compact operators are bounded*

Proposition 143 *Every bounded linear operator with finite rank is compact*

Definition 144 *Given X and Y Banach, we denote the space of all compact operators from X to Y by $\mathcal{K}(X, Y)$*

Proposition 145 *$\mathcal{K}(X, Y)$ is closed in $B(X, Y)$: Limits of compact operators are compact*

Theorem 146 *Let X be a Banach space, and let $T \in \mathcal{K}(X)$*

1. *For each $\epsilon > 0$, the number of eigenvalues λ of T with $|\lambda| \geq \epsilon$ is finite. In particular, T has countably many eigenvalues*
2. *if $\lambda \neq 0$ is an eigenvalue of T, then $\dim(\ker(T - \lambda)) < \infty$*
3. *If $\dim X = \infty$, then $0 \in \sigma(T)$*

10.1 The spectral theorem for compact selfadjoint operators

Proposition 147 *Let H be a complex Hilbert space, and let $T : H \to H$ be bounded. Then, T is selfadjoint if, and only if, $\langle Tx, x\rangle \in \mathbb{R}$ for all $x \in H$. As a consequence, all eigenvalues of selfadjoint operator are real*

Definition 148 *Let H be Hilbert space. A bounded linear operator $T : H \to H$ is nonnegative if $\langle Tx, x\rangle \geq 0$ for all $x \in H$. We shall write $T \geq 0$*

Proposition 149 *Let $T \geq 0$*

1. *T is selfadjoint*
2. *$\|Tx\|^2 \leq \|T\| \langle Tx, x\rangle$ for all $x \in H$*

Lemma 150 *Let H be a Hilbert space and let $T : H \to H$ be bounded and selfadjoint. Define*

$$a := \inf_{\|x\|=1} \langle Tx, x\rangle \qquad \text{and} \qquad b := \sup_{\|x\|=1} \langle Tx, x\rangle$$

we have the following

1. *$T - a \geq 0$ and $b - T \geq 0$*
2. *$\sigma(T) \subset [a, b]$, and $a, b \in \sigma(T)$*
3. *$\|T\| = \sup_{\|x\|=1} |\langle Tx, x\rangle| = \max\{|a|, |b|\}$*

Proposition 151 *Let H be a Hilbert space, and let $T : H \to H$ be a compact and selfadjoint. Then, either $-\|T\|$ or $\|T\|$ is an eigenvalue of T*

Corollary 152 *If $\sigma(T) = \{0\}$, then $T \equiv 0$*

Theorem 153 (The spectral theorem) *Let H be a separable Hilbert space, and let $T : H \to H$ be compact and selfadjoint. Then, there is an orthonormal basis of H composed of eigenvectors of T. More precisely, there exist countably many orthonormal eigenvectors $(e_n)_{n\in\mathcal{N}}$, corresponding to real eigenvalues $(\lambda_n)_{n\in\mathcal{N}}$, such that*

$$Tx = \sum_{n\in\mathcal{N}} \lambda_n \langle x, e_n\rangle e_n$$

for all $x \in H$. If $\dim(H) = \infty$, then $\mathcal{N} = \mathbb{N}$ and $\lim_{n\to\infty} \lambda_n = 0$